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On p-regular G-conjugacy class sizes

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Abstract

Let N be a normal subgroup of a p-solvable group G. The purpose of this paper is to investigate some properties of N under the condition that the two longest sizes of the non-central p-regular G-conjugacy classes of N are coprime. Some known results are generalized.

MSC: 20E45

Keywords: normal subgroups; G-conjugacy class sizes; p-regular elements

1 Introduction

All groups considered in this paper are finite. Let G be a group, and x an element of G. We denote by x^G the conjugacy class of G containing x and by $|x^G|$ the size of x^G.

The relationship between the p-regular conjugacy class sizes and the structure of a group G has been studied by many authors, see, for example, [1–5]. Let N be a normal subgroup of a group G. Clearly N is the union of some conjugacy classes of G. So, it is interesting to decide the structure of N by some arithmetical properties of the G-conjugacy class contained in N, for example, [3, 6, 7]. Particularly, in [3], we decided the structure of N when N possesses two G-conjugate class sizes. In this paper, the case considered is that N has more than two G-conjugate class sizes.

In a recent paper [1], the authors studied the structure of G under the condition that the largest two p-regular conjugacy class sizes (say, m and n) of G are coprime, where $m > n$ and $p \nmid n$. Notice that, when $G = N$, the condition n dividing $|N/Z(N)|$ is of course true, so our aim is, by eliminating the assumption $p \nmid n$, to investigate the properties of N under the corresponding condition. More precisely, we prove the following.

Theorem A *Let N be a normal subgroup of a p-solvable group G. If $m = |b^G| > |a^G| = n$ are the two longest sizes of the non-central p-regular G-conjugacy classes of N with $(m, n) = 1$ and n dividing $|N/Z(N)|$, where $a, b \in N$, then either a p-complement of $N/Z(N)$ is a prime power order group or*

(i) $|N|_{p'} = |a^N|_{p'}|b^N|_{p'}|Z(N)|_{p'}$;
(ii) *$|x^G| = m$ for any non-central p-regular element $x \in C_N(b)$. Furthermore, $C_N(b)_{p'}$ is abelian;*
(iii) *if d and t are two non-central p-regular elements of N such that $|t^G| \neq m = |d^G|$, then $(C_N(t) \cap C_N(d))_{p'} = Z(N)_{p'} \leq Z(G)$ and $n_{p'}$ divides $|t^N|$.*

Based on this, in Section 4, we give our improvement and generalization of [1] by considering the case that p does not divide n.

Let π be a set of some primes; we use x_π and $x_{\pi'}$ for the π-component and the π'-component of x, respectively. Moreover, G_π denotes a Hall π-subgroup of G, $G_{\pi'}$ a Hall π'-subgroup of G, n_π the π-part of n whenever n is a positive integer. Apart from these, we call an element x non-central if $x \notin Z(G)$, where $Z(G)$ is the center of G. Following [8] a group G is said to be quasi-Frobenius if $G/Z(G)$ is a Frobenius group and then the inverse images of the kernel and a complement of $G/Z(G)$ are called the kernel and complement of G.

2 Preliminaries

We first list some lemmas which are useful in the proof of our main result.

Lemma 2.1 [9, Lemma 1.1] *Let N be a normal subgroup of a group G and x an element of G. Then*:

(a) $|x^N|$ *divides* $|x^G|$;

(b) $|(Nx)^{G/N}|$ *divides* $|x^G|$.

Lemma 2.2 *Let N be a p-solvable normal subgroup of a group G and $B = b^G$, $C = c^G$ with $(|B|, |C|) = 1$, where b, c are two p-regular elements of N. Then*:

(a) $G = C_G(b)C_G(c)$.

(b) *$BC = CB$ is a p-regular G-conjugacy class of N and $|BC|$ divides $|B||C|$.*

Proof Set $G = N$ in Lemma 1 of [4], the proof is finished. □

Lemma 2.3 *Let N be a p-solvable normal subgroup of a group G and B_0 be a non-central p-regular G-conjugacy class of N with the largest size. Then the following properties hold*:

(a) *Let C be a p-regular G-conjugacy class of N with $(|B_0|, |C|) = 1$, then $|\langle C^{-1}C\rangle|$ divides $|B_0|$.*

(b) *Let $n, m = |B_0|$ be two largest p-regular G-conjugacy class sizes of N with $(m, n) = 1$ and D be a p-regular G-conjugacy class of N with $|D| > 1$. If $(|D|, n) = 1$, then $|D| = m$.*

Proof (a) By Lemma 2.2(b), CB_0 is a p-regular G-conjugacy class of N. Clearly, $|CB_0| \geq |B_0|$, so the hypotheses of the lemma imply that $|CB_0| = |B_0|$, from which it follows that $C^{-1}CB_0 = B_0$, and hence $\langle C^{-1}C\rangle B_0 = B_0$, consequently $|\langle C^{-1}C\rangle|$ divides $|B_0|$.

(b) Suppose that A is a p-regular G-conjugacy class and $|A| = n$. Lemma 2.2(b) implies that DA is a p-regular G-conjugacy class. Also $|DA| \geq |A|$, so $|DA| = n$ or m. If $|DA| = n$, then $D^{-1}DA$ is a p-regular G-conjugacy class, and hence $D^{-1}DA = A$, which implies that $\langle D^{-1}D\rangle A = A$. It follows that $|\langle D^{-1}D\rangle|$ divides $|A|$. On the other hand, $\langle D^{-1}D\rangle \subseteq \langle A^{-1}A\rangle$, so $|\langle D^{-1}D\rangle|$ divides $|\langle A^{-1}A\rangle|$. By (a), we find that $|\langle A^{-1}A\rangle|$ divides $|B_0|$, from which it follows that $|\langle D^{-1}D\rangle|$ divides $|B_0|$, a contradiction. Consequently $|DA| = m$, equivalently, $|B_0|$ divides $|A||D|$, it follows that $|D| = |B_0|$ by the hypotheses of the lemma, as wanted. □

Lemma 2.4 *Suppose that N is a p-solvable normal subgroup of a group G, Let B_0 be a non-central p-regular G-conjugacy class of N with the largest size. Write*

$$M = \big\langle D | D \text{ is a } p\text{-regular } G\text{-conjugacy class of } N \text{ with } \big(|D|, |B_0|\big) = 1\big\rangle.$$

Then $M_{p'}$ is abelian, furthermore, if $(Z(G)\cap N)_{p'} < M_{p'}$, then $\pi(M_{p'}/(Z(G)\cap N)_{p'}) \subseteq \pi(B_0)$.

Proof Write

$$K = \langle D^{-1}D \mid D \text{ is a } p\text{-regular } G\text{-conjugacy class of } N \text{ with } (|D|, |B_0|) = 1\rangle.$$

By the definition of M and K, we have $K = [M, G]$. Let $d \in D$, where D is a p-regular G-conjugacy class of N with $(|D|, |B_0|) = 1$. Applying Lemma 2.3(a), we have $\pi(K) \subseteq \pi(B_0)$, which implies that $(|K|, |D|) = 1$, hence $|d^K| = 1$. It shows that $K = C_K(d)$, so $K \leq Z(M)$. Notice that $M/K \leq Z(G/K)$, we find that M is nilpotent, hence $M = P \times M_{p'}$, where $P \in \mathrm{Syl}_p(M)$. Obviously, $(Z(G) \cap N)_{p'} \leq M_{p'}$. If $(Z(G) \cap N)_{p'} < M_{p'}$, let $r \in \pi(M_{p'}/(Z(G) \cap N)_{p'})$, $R \in \mathrm{Syl}_r(M)$, then $R \trianglelefteq G$. Also $1 \neq [R, G] \leq [M, G] = K$, we have $r \in \pi(K) \subseteq \pi(B_0)$, which implies that $\pi(M_{p'}/(Z(G) \cap N)_{p'}) \subseteq \pi(B_0)$. Suppose that D is a generating class of M and $d \in D$, then $|d^R|$ divides $|R|$, and $|d^R|$ divides $|D|$. The fact that $(|R|, |D|) = 1$ implies that $R = C_R(d)$, so $R \leq Z(M)$ by the nilpotence of M, which shows that $M_{p'}$ is abelian. □

3 Proof of Theorem A

In this section we are equipped to prove the main result.

Proof of Theorem A Suppose that $N/Z(N)$ is not a prime power order group. We will complete the proof by the following steps:

Step 1 *We may assume that $N_r \nleq Z(G)$ for every p'-prime factor r of $|N|$.*

Otherwise, there exists a p'-prime factor r of $|N|$ such that $N_r \leq Z(G)$, then $N = N_{r'} \times N_r$. Obviously, $N_{r'}$ satisfies the condition of the theorem. Application of the induction hypothesis to $|N|$ shows that the conclusion of the theorem holds, and hence we may assume that $N_r \nleq Z(G)$ for every p'-prime factor r of $|N|$.

Step 2 *If the p-regular element $x \in Z(C_G(b)) \cap N$, then either $x \in Z(G)$, or $C_G(x) = C_G(b)$.*

Obviously, $C_G(b) \leq C_G(x)$, which implies that $|x^G|$ divides $|b^G|$, it follows that $(|x^G|, n) = 1$. If $x \notin Z(G)$, Lemma 2.3 shows that $|x^G| = m$, so $C_G(x) = C_G(b)$.

Step 3 *We may assume that b is a prime power order q-element ($q \neq p$).*

Let q be a prime factor of $o(b)$, b_q be the q-component and $C_G(b_q) \neq G$. Notice that $C_G(b) = C_G(b_q b_{q'}) = C_G(b_q) \cap C_G(b_{q'}) \subseteq C_G(b_q)$, applying Step 2, we have $C_G(b_q) = C_G(b)$, and this completes the proof by replacing b with b_q.

Step 4 *$C_N(b) = P_b Q_b \times L$, where P_b is a Sylow p-subgroup of $C_N(b)$, Q_b is a Sylow q-subgroup of $C_N(b)$, L is a $\{p, q\}'$-Hall subgroup of $C_N(b)$ with $L \leq Z(C_G(b))$. Particularly, if $L \nleq Z(G)$, then $C_N(b)_{p'} \leq Z(C_G(b))$.*

Let $x \in C_N(b)$ be a $\{p, q\}'$-element. Notice that $C_G(bx) = C_G(b) \cap C_G(x) \leq C_G(b)$, we find that $|b^G|$ divides $|(bx)^G|$, so the maximality of $|b^G|$ implies that $|b^G| = |(bx)^G|$, and hence $C_G(bx) = C_G(b) \leq C_G(x)$, from which it follows that $x \in Z(C_G(b))$. Consequently, $C_N(b) = P_b Q_b \times L$, where P_b is a Sylow p-subgroup of $C_N(b)$, Q_b is a Sylow q-subgroup of $C_N(b)$, L is a $\{p, q\}'$-Hall subgroup of $C_N(b)$ with $L \leq Z(C_G(b))$.

Particularly, if $L \nleq Z(G)$, let $y \in L$ be a non-central prime power order r-element, then Step 2 implies that $|y^G| = m$, and hence $C_G(y) = C_G(b)$. By the above argument, we have $C_N(y) = P_yR_y \times L_y$, where P_y is a Sylow p-subgroup of $C_N(b)$, R_y is a Sylow r-subgroup of $C_N(b)$, L_y is a $\{p,r\}'$-Hall subgroup of $C_N(b)$ with $L_y \leq Z(C_G(b))$. Clearly, $C_N(b) = P_b \times LL_y$, so $C_N(b)_{p'} = LL_y \leq Z(C_G(b))$.

Step 5 $q \nmid m$.

If $q|m$, then $q \nmid n$, and of course we have $q \nmid |a^N|$ by Lemma 2.1. Notice that $|a^N| = |N : C_N(a)| = |C_N(b) : C_N(a) \cap C_N(b)|$, hence $C_N(a) \cap C_N(b)$ contains a Sylow q-subgroup of $C_N(b)$, which implies that $b \in C_N(a) \cap C_N(b)$, therefore $a \in C_N(b)$. We distinguish two cases according to the structure of $C_N(b)$.

(1) If $L \nleq Z(G)$, Step 4 implies that $C_N(b)_{p'} \leq Z(C_G(b))$. By the above $a \in C_N(b)$, we have $C_G(b) \leq C_G(a)$, which leads to $|a^G|$ dividing $|b^G|$, a contradiction.

(2) If $L \leq Z(G)$, then we may assume that a is a q-element since $a \in C_N(b) = P_bQ_b \times L$. For every $\{p,q\}'$-element $x \in C_N(a)$, we have $C_G(ax) = C_G(a) \cap C_G(x) \leq C_G(a)$, the hypothesis of the theorem shows that $C_G(ax) = C_G(a) \leq C_G(x)$, from which it follows that $x \in Z(C_G(a))$. Notice that $b \in C_N(a)$, so $x \in C_N(b)$, which implies that $x \in L$, consequently a p-complement of $N/Z(N)$ is a prime power order group, a contradiction.

Step 6 *We may assume that a is a $\{p,q\}'$-element.*

Let $a = a_qa_{q'}$, where a_q, $a_{q'}$ are the q-component and q'-component of a, respectively. Notice that $C_G(a) = C_G(a_q) \cap C_G(a'_q) \subseteq C_G(a_q)$; we have $a_q \in M$, where M is ever defined in Lemma 2.4. If $a_q \notin Z(G)$, then, by Lemma 2.4, $q \in \pi(M_{p'}/(Z(G) \cap N)_{p'}) \subseteq \pi(m)$, in contradiction to Step 5.

Step 7 $C_N(a)_q \leq Z(G)_q$.

Suppose that there exists a non-central q-element $y \in C_N(a)$, then $C_G(ay) = C_G(a) \cap C_G(y) \subseteq C_G(a)$, so we have $C_G(ay) = C_G(a)$ by the hypothesis of the theorem. It follows that $ay \in M$, which implies that $y \in M$ because of $a \in M$. So $q \in \pi(M_{p'}/(Z(G) \cap N)_{p'}) \subseteq \pi(m)$, in contradiction to Step 5. Hence, $C_N(a)_q \leq Z(G)_q$, as required.

Step 8

(8.1) $(C_N(a) \cap C_N(b))_{p'} = Z(N)_{p'} \leq Z(G)$.

(8.2) $|N|_{p'} = |a^N|_{p'}|b^N|_{p'}|Z(N)|_{p'}$.

(8.1) Our immediate object is to show that $(C_N(a) \cap C_N(b))_{p'} \leq Z(G)$. Otherwise, then there exists a non-central p-regular element $y \in C_N(a) \cap C_N(b)$. In view of Step 4, if $L \leq Z(G)$, then we may assume that y is a q-element. So the q-element y lies in $C_N(a)$, in contradiction to Step 7. So another possibility $L \nleq Z(G)$ must prevail, which implies that $y \in C_N(b)_{p'}$. Keeping in mind that $C_N(b)_{p'} \leq Z(C_G(b))$, we have $C_G(y) = C_G(b)$ by the hypotheses, from which it follows that $a \in C_N(y) = C_N(b)$. Applying $C_N(b)_{p'} \leq Z(C_G(b))$ once again, we have $C_G(b) \leq C_G(a)$, a contradiction, as wanted. Consequently, $(C_N(a) \cap C_N(b))_{p'} \leq (Z(G) \cap N)_{p'}$. Notice that $(Z(G) \cap N)_{p'} \leq Z(N)_{p'}$ and $Z(N)_{p'} \leq (C_N(a) \cap C_N(b))_{p'}$, so $(C_N(a) \cap C_N(b))_{p'} = Z(N)_{p'}$, as required.

(8.2) Next, the conclusion $|N|_{p'} = |a^N|_{p'}|b^N|_{p'}|Z(N)|_{p'}$ is to be dealt with. Obviously, $(|a^N|, |b^N|) = 1$ by Lemma 2.1 in terms of $(|a^G|, |b^G|) = 1$, which implies that $N = C_N(a) \times C_N(b)$. This leads to $|N| = |C_N(a)||C_N(b)|/|C_N(a) \cap C_N(b)|$, and hence $|N| = |a^N||b^N| \times |C_N(a) \cap C_N(b)|$. Notice that $(C_N(a) \cap C_N(b))_{p'} = Z(N)_{p'}$, and we have $|N|_{p'} = |a^N|_{p'}|b^N|_{p'} \times |Z(N)|_{p'}$, as wanted.

Step 9 *It followed that $n = |a^N|p^\alpha$, where $\alpha \geq 0$. And hence, if $L \leq Z(G)$, $|a^G|$ is at most a $\{p, q\}$-number.*

Notice that $|N| = |a^N||b^N||C_N(a) \cap C_N(b)|$ and $(C_N(a) \cap C_N(b))_{p'} = Z(N)_{p'}$, and by the hypothesis that n divides $|N/Z(N)|$, we have $n = |a^N|p^\alpha$, where $\alpha \geq 0$. On the other hand, $|a^N| = |C_N(b)|/|C_N(a) \cap C_N(b)|$. By what has already been proved, we find that, if $L \leq Z(G)$, then $|a^N|$ is at most a $\{p, q\}$-number, and so is $|a^G|$.

Step 10 *$|x^G| = m$ for any non-central p-regular element $x \in C_N(b)$, and therefore $C_N(b)_{p'}$ is abelian.*

By the structure of $C_N(b)$, we distinguish two cases:

(1) If $L \nleq Z(G)$, then $C_N(b)_{p'} \leq Z(C_G(b))$ by Step 4. Obviously, $C_N(b)_{p'}$ is abelian. In addition, for any non-central p-regular element $x \in C_N(b)$, we find that $|x^G|$ divides $|b^G|$, application of Lemma 2.3 yields $|x^G| = m$.

(2) If $L \leq Z(G)$, by Step 9, we know $|a^G|$ is at most a $\{p, q\}$-number. Also, $x \in C_N(b) = P_bQ_b \times L$ is a non-central p-regular element; we may assume that $x \in Q_b \setminus Z(G)$ if necessary by a suitable conjugate.

On the other hand, we know $M_{p'}$ is abelian and $Z(G)_q \cap N \leq M_{p'} \leq C_N(a)$, by Step 7, we can write $M_{p'} = S \times (Z(G)_q \cap N)$ where $q \nmid |S|$. Notice that $\langle x \rangle$ acts coprimely on the abelian subgroup S, and by coprime action properties, we have

$$S = \left[S, \langle x \rangle\right] \times C_S(x).$$

Denote by $U = [S, \langle x \rangle]$. As $a \in S$, we can write $a = uw$ with $u \in U$, $w \in C_S(x)$. Consider the element $g = wx$; we have

$$C_G(g) = C_G(w) \cap C_G(x) \leq C_G(x).$$

If $|g^G| = m$, then $|x^G| = m$ by Lemma 2.3 since $|x^G|$ divides $|g^G|$.

If $|g^G| = n$, notice that $|x^G|$ divides $|g^G|$, then $x \in M_q$. However, $M_q \leq C_N(a)_q \leq Z(G)_q$, a contradiction.

So we are left with only one alternative: $|g^G| < n$. Keeping in mind that G is a p-solvable group, let T be a Hall $\{p, q\}$-subgroup of G, ST is a subgroup of G since S is a normal subgroup of G. Notice that

$$\begin{aligned} \left|ST : C_{ST}(g)\right| \leq \left|g^G\right| < n = \left|a^G\right| \\ &= \left|G : C_G(a)\right| \\ &= \left|G : C_G(a)\right|_{\{p,q\}} \\ &= |T| : \left|C_G(a)\right|_{\{p,q\}}. \end{aligned} \tag{3.1}$$

Moreover, since $S \trianglelefteq G$, S is abelian, $S \cap T = 1$, and $S \leq C_G(w)$ we have

$$\begin{aligned} C_{ST}(g) &= C_{ST}(w) \cap C_{ST}(x) \\ &= SC_T(w) \cap C_S(x)C_T(x) \\ &= C_S(x)\bigl[SC_T(w) \cap C_T(x)\bigr] \\ &= C_S(x)\bigl[C_T(w) \cap C_T(x)\bigr]. \end{aligned} \tag{3.2}$$

We denote $D = C_T(w) \cap C_T(x)$. Combining (3.1) and (3.2), we have

$$\frac{|T|}{|C_G(a)|_{\{p,q\}}} > \frac{|S||T|}{|C_S(x)||D|}.$$

This implies that $|D| : |C_G(a)|_{\{p,q\}} > |S : C_S(x)| = |U|$.

On the other hand, as $D \leq C_G(x)$ and $U = [S, \langle x\rangle]$, then D normalizes U. Also,

$$C_D(u) = C_G(u) \cap D = C_T(u) \cap C_T(w) \cap C_T(x) \leq C_T(a) \cap C_T(x),$$

so

$$\bigl|C_D(u)\bigr| \leq \bigl|C_T(a) \cap C_T(x)\bigr| \leq \bigl|C_G(a)\bigr|_{\{p,q\}}.$$

Therefore

$$\bigl|u^D\bigr| = \bigl|D : C_D(u)\bigr| = |D| : \bigl|C_D(u)\bigr| \geq |D| : \bigl|C_G(a)\bigr|_{\{p,q\}} > |U|,$$

a contradiction.

So the argument on the above three cases forces $|x^G| = m$.

Next, we show that $(C_N(b))_{p'}$ is abelian when $L \leq Z(G)$. Notice that $C_N(b) = P_bQ_b \times L$, it is enough to show that Q_b is abelian. For any non-central element $x \in Q_b$, we have $C_S(x) \leq Z(G)_{p'}$. Otherwise, by the above, we have $|x^G| = m$. Replacing b with x, we have $(C_N(a) \cap C_N(b))_{p'} \nleq Z(G)$, which contradicts Step 8. Therefore $Q_b/Q_b \cap Z(G)$ acts on the group $S/S \cap Z(G)$ fixed-point-freely, which implies that $Q_b/Q_b \cap Z(G)$ is cyclic or a generalized quaternion. So, if Q_b is not abelian, then $q = 2$ and $Q_b/Q_b \cap Z(G)$ is a generalized quaternion group. Now $b \in Z(Q_b)$ but $b \notin Z(G)$, so there exists an element $y \in Q_b$ and $y \notin Z(Q_b)$ such that $b = y^2c$ where $c \in Z(G) \cap Q_b$. So $C_G(y) \leq C_G(b)$, which indicates that $y \in Z(C_G(b))$, and of course we have $y \in Z(Q_b)$, a contradiction. Therefore Q_b is abelian, as required.

Step 11 *If d and t are two non-central p-regular elements of N such that $|t^G| \neq m = |d^G|$, then*:

(11.1) $(C_N(t) \cap C_N(d))_{p'} = Z(N)_{p'} \leq Z(G)$.

(11.2) $|a^N|_{p'}$ *divides* $|t^N|$, *and hence* $n_{p'}$ *divides* $|t^N|$.

(11.1) Firstly, it will be established that $(C_N(t) \cap C_N(d))_{p'} \leq Z(G)$. Otherwise, there exists a non-central p-regular element $y \in C_N(t) \cap C_N(d)$, and we will distinguish two cases as for the structure of $C_N(d)$.

(1) If $L \nleq Z(G)$, then $C_N(d)_{p'} \leq Z(C_G(d))$ by Step 4, which implies that $y \in Z(C_G(d))$. However, y is non-central, the hypotheses of the theorem shows that $C_G(y) = C_G(d)$, from which it follows that $t \in C_N(d)_{p'}$. Notice that $C_N(d)_{p'} \leq Z(C_G(d))$, we have $C_G(t) = C_G(d)$ by the hypotheses of the theorem, but this contradicts the fact that $|t^G| \neq m = |d^G|$. Therefore $(C_N(t) \cap C_N(d))_{p'} \leq Z(G)$.

(2) Suppose that $L \leq Z(G)$. Our immediate object is to show that t can be assumed to be a q'-element. In fact, let t_q be the q-component of t. If $t_q \notin Z(G)$, Step 10 implies that $|{t_q}^G| = m$. Notice that $C_G(t) \leq C_G(t_q)$, we have $|t^G| = m$ by Lemma 2.3, against the fact that $|t^G| \neq m = |d^G|$. It follows that $t_q \in Z(G)$, so $C_G(t) = C_G(t_{q'})$, where $t_{q'}$ is the q'-component of t. Thus, without loss of generality, we may assume that t is a q'-element.

Also, in this case, we may assume that y is a q-element. Keeping in mind that $q \nmid m$, application of Step 10 once again, we have $|y^G| = m$. Moreover,

$$C_G(ty) = C_G(t) \cap C_G(y) \subseteq C_G(y),$$

which implies that $|(ty)^G| = m$ by the maximality of m. So $C_G(ty) = C_G(y) \leq C_G(t)$, which shows that $|t^G| = m$ by Lemma 2.3(b), a contradiction. Thus, $(C_N(d) \cap C_N(t))_{p'} \leq Z(G)$.

Next, in a similar manner as in Step (8.1), the equality in (11.1) follows.

(11.2) Consider the quotient group $(C_N(b)/Z(N))_{p'}$ and the set $\{t^N\}$. For any $\bar{x} \in (C_N(b)/Z(N))_{p'}$ and $y \in \{t^N\}$, without loss of generality, we may assume that $\bar{x} = xZ(N)$ where $x \in C_N(b)$, and we set

$$y^{\bar{x}} = y^x.$$

Clearly, $(C_N(b)/Z(N))_{p'}$ acts as a group on the set $\{t^N\}$ through the above action. Obviously, $t^N \cap C_N(b) = \emptyset$ and $(C_N(t) \cap C_N(b))_{p'} = Z(N)_{p'}$, this shows that the group $(C_N(b)/Z(N))_{p'}$ acts on the set $\{t^N\}$ fixed-point-freely. Therefore $|C_N(b)/Z(N)|_{p'}$ divides $|t^N|$. Notice that $|C_N(b)/Z(N)|_{p'} = |a^N|_{p'}$, so we find that $|a^N|_{p'}$ divides $|t^N|$. By Step 9, $n_{p'}$ divides $|t^N|$, which is fairly straightforward. □

Corollary 1 *Suppose that G is a group. Let $|b^G| = m > n = |a^G|$ be the two longest sizes of the non-central conjugacy classes of G. If $(m,n) = 1$, then G is solvable and the conjugacy class size of the element in G is exactly 1, n or m.*

Proof Let p be a prime and p be not a prime factor of $|G|$, then G is p-solvable. Obviously, n divides $|G/Z(G)|$. So, in Theorem A, by taking $N = G$, we have $|x^G| = n_{p'} = n$ if $|x^G| \neq m$, from which it follows that $x \in M_{p'}$. On the other hand, $|y^G| = n$ for any non-central element $y \in C_G(a)$. In fact, if $|y^G| = m$, we have $y \in C_G(a) \cap C_G(y)$, in contradiction to (iii) in Theorem A. It follows that $C_G(a) = M_{p'}$ is abelian. By Theorem A, $C_G(b)$ is abelian, and the solvability of G is obtained. □

4 Application of Theorem A

Based on Theorem A, we consider the case that p does not divide n.

Theorem B *Let N be a normal subgroup of a p-solvable group G. If $m = |b^G| > |a^G| = n$ are the two longest sizes of the non-central p-regular G-conjugacy classes of N with $(m,n) = 1$*

and n dividing $|N/Z(N)|$, where $a, b \in N$ and $p \nmid n$, then either a p-complement of $N/Z(N)$ is a prime power order group or

(i) *the p-regular G-conjugacy class sizes of N are exactly* 1, n *and* m.

(ii) *Let x be a non-central p-regular element of N. If $x \in C_N(a)$, then $|x^G| = n$; if $x \in C_N(b)$, then $|x^G| = m$.*

(iii) *A p-complement of N is a solvable quasi-Frobenius group with abelian kernel and complement.*

(iv) *The conjugacy class sizes of a p-complement of N are exactly* 1, $|a^N|$, *and* $|b^N|_{p'}$.

Proof Suppose that a p-complement of $N/Z(N)$ is not a prime power order subgroup, and t, d are two non-central p-regular elements of N with $|t^G| \neq m = |d^G|$.

(i) Now, we show the G-conjugacy class size of the p-regular element of N is 1, n or m. Obviously, $|t^G| = n$ is to be dealt with. Since n is a p'-number, by Theorem A, we find that n divides $|t^G|$, forcing $|t^G| = n$, as wanted.

(ii) Let $x \in C_N(a)$ be a non-central p-regular element. If $|x^G| = m$, then $x \in (C_N(a) \cap C_N(x))_{p'}$, in contradiction to (iii) in Theorem A. So $|x^G| = n$. By Theorem A once again, the conclusion (ii) is obtained.

(iii) Let $C_N(a)_{p'}C_N(b)_{p'} = H$, then $H = M_{p'}C_N(b)_{p'}$ is a p-complement of N. Now, $C_N(a)_{p'}$ ($= M_{p'}$) and $C_N(b)_{p'}$ are abelian, we find that H is solvable. Taking into account $(C_N(a) \cap C_N(b))_{p'} = Z(N)_{p'}$ by Theorem A, we find that $Z(H) = Z(N)_{p'}$. For convenience, we employ '–' to work in the factor group modulo $Z(H)$. Notice that $|\overline{M_{p'}}| = |\overline{C_N(a)_{p'}}|$ divides $|b^N|$ and $|\overline{C_N(b)_{p'}}|$ divides $|a^N|$, so we have $(|\overline{C_N(b)_{p'}}|, |\overline{M_{p'}}|) = 1$.

Now, $\overline{M_{p'}}$ is an abelian normal subgroup of $\overline{H}$. To prove that $\overline{H}$ is a Frobenius group, we are left to show $C_{\overline{M_{p'}}}(\bar{x}) = 1$ for any non-central element $x \in C_N(b)_{p'}$. Otherwise, there exists a non-central element $y \in M_{p'}$ such that $1 \neq \bar{y} \in C_{\overline{M_{p'}}}(\bar{x})$, then $(\overline{xy})^{o(\bar{x})} = (\bar{x}\bar{y})^{o(\bar{x})} = \bar{y}^{o(\bar{x})} \neq 1$ since $(o(\bar{x}), o(\bar{y})) = 1$. Hence

$$\overline{xy}^{o(\bar{x})} = \bar{y}^{o(\bar{x})} \in \overline{C_N(x)} \cap \overline{C_N(xy)} = \overline{C_N(x) \cap C_N(xy)} = \overline{C_N(x) \cap C_N(y)},$$

so $C_N(x) \cap C_N(y)$ contains a non-central p-regular element.

Notice that $|x^G| = m$ and $|y^G| = n$; by Theorem A, we have $(C_N(x) \cap C_N(y))_{p'} \leq Z(G)$, in contradiction to the previous paragraph. So a p-complement of N is a solvable quasi-Frobenius group with the abelian kernel $M_{p'}$ and complement $C_N(b)_{p'}$.

(iv) Let $x \in H$ be a non-central element. We have $|x^G| = n$ or m by (i). If $|x^G| = n$, then

$$\left|x^H\right| = \left|H : C_H(x)\right| = \left|C_N(b)_{p'}/Z(N)_{p'}\right| = \left|C_N(b)/C_N(a) \cap C_N(b)\right| = \left|a^N\right|.$$

If $|x^G| = m$, then $C_N(x)_{p'}M_{p'}$ is also a p-complement of N. In view of the p-solvability of N, replacing H if necessary by a suitable conjugate, we may assume that $C_N(x)_{p'} \leq H$. Notice that $C_N(x)_{p'} \cap M_{p'} \leq (C_N(x) \cap C_N(a))_{p'} = Z(N)_{p'}$, we have $|C_N(x)_{p'}| = |C_N(b)_{p'}|$. Also,

$$C_N(b)_{p'}/Z(N)_{p'} = C_N(b)_{p'}/Z(H) = \overline{C_N(b)_{p'}},$$

keeping in mind that $(|\overline{C_N(b)_{p'}}|, |\overline{M_{p'}}|) = 1$, by the solvability of H, we find that $\overline{C_N(x)_{p'}}$ and $\overline{C_N(b)_{p'}}$ are conjugate in $\overline{H}$. Consequently, we may assume that there exists an element

$g \in N$ such that $x^g \in C_N(b)$, so

$$\begin{aligned} |x^H| = |x^{gH}| = |H : C_H(x^g)| &= |C_N(a)_{p'}/Z(N)_{p'}| \\ &= |C_N(a)/C_N(a) \cap C_N(b)|_{p'} \\ &= |b^N|_{p'}. \end{aligned}$$

So the conjugacy class sizes of H are 1, $|a^N|$, and $|b^N|_{p'}$. □

Corollary 2 [1, Theorem A] *Suppose that G is a p-solvable group. Let* $m > n > 1$ *be the two longest sizes of the non-central p-regular conjugacy classes of G. Suppose that* $(m, n) = 1$ *and p is not a prime divisor of n. Then G is solvable and*

(a) *the p-regular conjugacy class sizes of G are exactly* 1, n, *and* m;
(b) *a p-complement of G is a quasi-Frobenius group with abelian kernel and complement. Furthermore, its conjugacy class sizes are exactly* 1, n, *and* $m_{p'}$.

Proof Obviously, n divides $|G/Z(G)|$. So, in Theorem B, by taking $N = G$, the proof of this corollary is finished. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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